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Projecting $(N - 1)$ -cycles to zero on hyperplanes in \mathbf{R}^{N+1}

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Abstract

The projection of a compact oriented submanifold $M^{n-1} \subset \mathbf{R}^{n+1}$ on a hyperplane P^n can fail to bound any region in P . We call this “projecting to zero.” Example: The equatorial $\mathbf{S}^1 \subset \mathbf{S}^2 \subset \mathbf{R}^3$ projects to zero in any plane containing the x_3 -axis. Using currents to make this precise, we show a lipschitz (homology) $(n - 1)$ -sphere embedded in a compact, strictly convex hypersurface cannot project to zero on $n + 1$ linearly independent hyperplanes in \mathbf{R}^{n+1} . We also show, using examples, that all the hypotheses in this statement are sharp.

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1. Introduction

Basic differential topology shows that a smooth compact submanifold M^{n-1} embedded in \mathbf{R}^n always bounds a domain. But when we embed M in \mathbf{R}^{n+1} , and then project it orthogonally into a hyperplane P^n ,

$$M^{n-1} \subset \mathbf{R}^{n+1} \xrightarrow{\pi} P^n$$

the projection $\pi(M)$ will typically bound a linear combination of simple domains in P with “winding number” coefficients. In certain non-generic circumstances all of these winding numbers can vanish, and in such cases, we will say that M *projects to zero on P* .

For instance, if we embed the round circle \mathbf{S}^1 into a horizontal plane in \mathbf{R}^3 , it projects to zero on any vertical 2-plane; the projection “cancels itself” by traversing a single line segment once in each direction (Fig. 1a).

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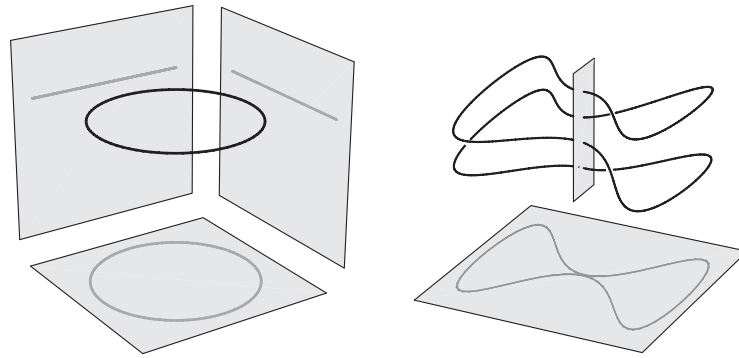


Fig. 1. (a) A horizontal circle projects to zero on vertical (but not horizontal) planes. (b) The projection of this embedded loop on the horizontal plane is immersed, yet vanishes.

Fig. 1b depicts a subtler—perhaps even surprising—way in which cancellation can occur. There, all winding numbers vanish, even though the projected loop is *immersed*—a pinched figure-of-eight with each lobe traversed once in both directions.

While examples like these are not hard to construct, we direct our efforts here toward the difficulties that arise when one tries to make a compact embedded $(n - 1)$ -manifold project to zero *simultaneously on a maximal collection of independent hyperplanes in \mathbf{R}^{n+1}* . We call a set of hyperplanes *independent* if their normals form a linearly independent set.

Our main result, Theorem 5.3 below, isolates sharp conditions that obstruct this kind of simultaneous null-projection. In its statement, a *strict C^2 ovaloid* means a compact, convex C^2 hypersurface with no vanishing principal curvatures.

Theorem 5.3. *A lipschitz embedding of a homology $(n - 1)$ -sphere on a strict C^2 ovaloid in \mathbf{R}^{n+1} cannot project to zero on $n + 1$ independent hyperplanes.*

To prove this, we need a more precise definition of “projection to zero,” and in Section 3, we employ the language of *currents* for that purpose. For now, however, we forego rigor, give some simple geometric examples that illustrate the problem, and justify the hypotheses in our theorem above.

2. Examples

The first three examples below highlight topological issues that arise in the problem we investigate.

Example 2.1 (Connectedness). Failing connectedness, we can make a compact embedded hypersurface project to zero on *any* finite collection of hyperplanes, no matter how numerous.

Indeed, suppose we have N hyperplanes P_1, P_2, \dots, P_N in \mathbf{R}^{n+1} , with corresponding unit normals v_1, v_2, \dots, v_N . Take any embedded, oriented submanifold M_0 , and recursively define, for $i = 1, 2, \dots, N$,

$$M_i = M_{i-1} \cup (-M_{i-1} + c_i v_i).$$

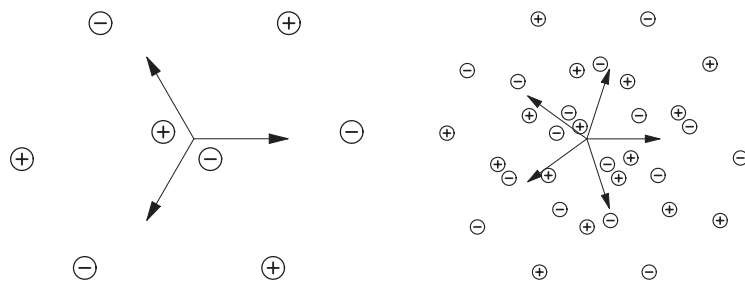


Fig. 2. (a) A collection of 2^3 signed points that projects to zero in 3 directions. (b) A collection of 2^5 points projecting to zero in 5 directions.

Here $-M_{i-1} + c_i v_i$ denotes an orientation-reversed copy of M_{i-1} , translated by c_i units in the v_i direction. By choosing each c_i large enough to make the union disjoint, we can preserve embeddedness throughout the construction.

Projection onto P_i annihilates v_i , so M_i projects to zero on P_i . Neither orientation-reversal, nor translation by v_{i+1} in the next step of the construction will destroy this property, so after N iterations, we obtain an embedded submanifold M_N that projects to zero on all the P_i 's, as claimed.

Fig. 2 illustrates this construction with $n=0$. The signed circles locate oriented points (0-currents) in the plane. Higher dimensions (or even higher codimensions) present no additional complications, and the construction clearly shows that without connectedness, nothing obstructs simultaneous projection to zero on any finite number of hyperplanes.

Example 2.2 (Embeddedness). Failing embeddedness, one can “fake” connectedness to exploit the same phenomenon just discussed. For instance, when $n=2$, the latitudinal circles $x_3 = \pm 1/2$ in S^2 , if oppositely oriented, satisfy all the hypotheses of Theorem 5.3 except the homological one. They violate the conclusion by projecting to zero on all $n+1=3$ coordinate planes.

The homological defect can be fixed at the expense of embeddedness, however: Connect these circles with a doubled longitudinal arc, parametrized once from the lower circle to the upper, and once from the upper to the lower. The doubled arc already cancels itself out in \mathbf{R}^3 , so the same holds for its projections. But the resulting connected curve—a lipschitz immersion of S^1 into the sphere—still projects to zero on all three coordinate planes (Fig. 3b).

Example 2.3 (Topological type). Theorem 5.3 specifies all the homology groups of M . To see that we need some such hypothesis beyond simply $H_0(M) = \{0\}$ (i.e., connectedness), consider the Clifford torus

$$T^2 := S^1 \times S^1 \subset \mathbf{R}^2 \times \mathbf{R}^2 \approx \mathbf{R}^4.$$

This torus clearly lies in the origin-centered sphere of radius $\sqrt{2}$ in \mathbf{R}^4 , which is certainly a strict C^2 ovaloid. Parametrizing T^2 by the map

$$X(u, v) = (\cos u, \sin u, \cos v, \sin v),$$

one immediately sees that its projection onto any of the $n+1=4$ coordinate 3-planes constitutes a cylinder $S^1 \times [-1, 1]$. But in each case, the parametrization traverses the interval $[-1, 1]$ once in

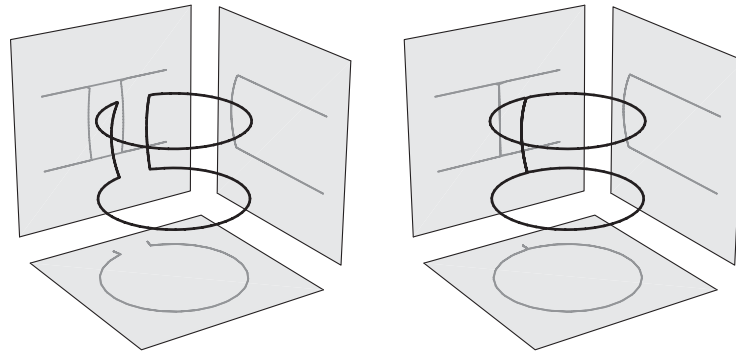


Fig. 3. (a) An embedded circle that lies on S^2 and projects to zero on 2 coordinate planes. (b) Closing the gap between the two longitudinal arcs, we make the loop project to zero on 3 independent planes, but lose embeddedness. (The horizontal projection vanishes by tracing the circle once each way.)

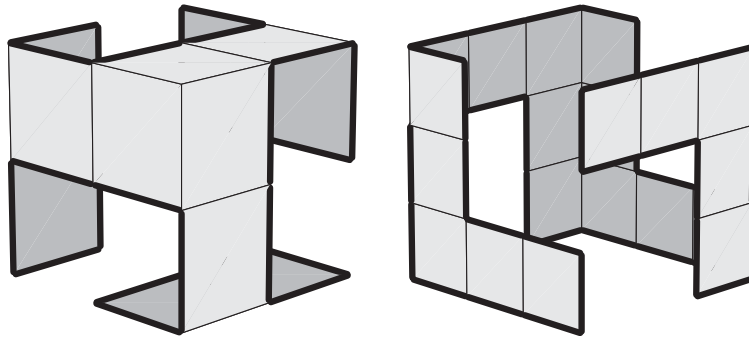


Fig. 4. The boundaries of these two polyhedra satisfy all hypotheses of Theorem 5.3 except for *strict* convexity of the surface—a cube—on which they lie. Both boundary loops project to zero on all 3 coordinate planes.

each direction, doubling the cylinder with opposite orientations. So all four coordinate projections vanish as currents.

Example 2.4 (Linear independence). The equator $x_{n+1} \equiv 0$ in S^n clearly projects to zero on all of the infinitely many hyperplanes that contain the x_{n+1} -axis. Since one can make n of these—but no more—linearly independent, our theorem cannot specify fewer projections than it does. Fig. 1a depicts the case $n = 2$.

Example 2.5 (Strict convexity). Finally, and perhaps most surprisingly, one cannot omit the *strict* convexity assumption; convexity alone does not suffice. Fig. 4 presents two loops—depicted as boundaries of polyhedral surfaces on the unit cube—which project to zero on all three standard coordinate planes.

The loops in Fig. 4 have corners, and one cannot simply round them off without destroying at least one of the null projections that make them interesting. For this reason, we originally guessed

that no *smooth* loop could project to zero in three independent directions, even without the strict convexity assumption in Theorem 5.3. Mohammad Ghomi has disproven this conjecture, however, by constructing an intricate counterexample on a “rounded” cube. His example appears as an appendix [4] to the present paper.

3. Lipschitz curves and currents

Definition 3.1. By an *oriented lipschitz k -chain* in a riemannian manifold N , we mean a lipschitz mapping $F: M^k \rightarrow N$, in which M is an oriented k -dimensional riemannian manifold. We call F a (compact) *k -cycle* when M is closed (compact).

Our main interest here lies with compact lipschitz $(n-1)$ -cycles in \mathbf{R}^{n+1} that *project to zero* on several hyperplanes, as discussed in the previous two sections. We now formulate that notion more precisely by viewing lipschitz k -chains as *k -dimensional currents*:

Definition 3.2 (Currents). Suppose N is a riemannian manifold. For the purposes of this paper, a *k -current* in N is simply a bounded linear function on the vector space $\mathcal{D}^k(N)$ comprising all compactly supported smooth differential k -forms on N .

This definition encompasses some very general objects, but the currents that interest us here all arise from oriented lipschitz k -chains in a very simple way. Indeed, any such chain $F: M \rightarrow N$ induces a k -current $[F]$, via

$$[F](\phi) := \int_M F^* \phi \quad \text{for all } \phi \in \mathcal{D}^k(N).$$

Since lipschitz mappings are differentiable almost everywhere (Rademacher’s Theorem [2, 3.1.6]), the integral makes sense. Further, the change-of-variable formula [2, 3.2.6], shows that composing F with any orientation-preserving diffeomorphism of M (i.e., “reparametrizing”) leaves $[F]$ unchanged. In this sense, the current $[F]$ is actually a more “geometric” object than the mapping F .

Currents also enjoy an elegant notion of boundary:

Definition 3.3 (Boundaries). The *boundary* of a k -current T in a manifold N is the $(k-1)$ -current ∂T characterized by

$$\partial T(\phi) := T(d\phi).$$

This is a very natural and geometric definition, because Stokes’ theorem combines with Definition 3.2 to show that whenever $F: M \rightarrow N$ is an immersed oriented submanifold with boundary, we have

$$\partial[F] = [F|_{\partial M}].$$

Definition 3.4 (Maps of currents). Given any k -current T in N , and a locally lipschitz mapping $G: N \rightarrow N'$ between riemannian manifolds, we get a new current $G_\# T$ in N' via

$$G_\# T := T \circ G^*.$$

Like the boundary operator, this notion of mapping is geometrically natural, in the sense that when $F:M \rightarrow N$ is an oriented lipschitz chain, we have $G_{\#}[F] = [F \circ G]$. Moreover, lipschitz mapping commutes with the boundary operator for currents, just as diffeomorphisms preserve boundaries of manifolds. More precisely, whenever $G:M \rightarrow N$ is a proper lipschitz map, and T is a k -current in M , we have

$$\partial(G_{\#}T) = G_{\#}(\partial T). \quad (1)$$

This fact is proven and discussed in [2, 4.1.14]; we shall need it more than once.

Finally, we consider the most basic relationship between currents and point-sets. The point-set associated with a mapping is its *image*. The analogous set associated with a current is its *support*:

Definition 3.5 (Supports). Suppose T is a current on a manifold N . We say that T *vanishes* (or $T \equiv 0$) on an open subset $U \subset N$ if $T(\phi) = 0$ for all 1-forms ϕ with support $\text{spt}(\phi) \subset U$. We then define $\text{spt}(T)$, the *support* of T , as the complement of the largest such open set. Equivalently,

$$\text{spt}(T) := \{p \in N : T \neq 0 \text{ on every neighborhood of } p\}.$$

Note that the support of a current is always closed.

We can now give precise meaning to our concept of “projection to zero”:

Definition 3.6 (Projection to zero). When $\pi:\mathbf{R}^{n+1} \rightarrow P$ is orthogonal projection onto an affine subspace P , and T is a current in \mathbf{R}^{n+1} such that $\pi_{\#}T \equiv 0$, we say that T *projects to zero on* P .

Example 3.7. Consider any smooth oriented k -cycle $F:M^k \rightarrow \mathbf{R}^{n+1}$. As geometric intuition suggests, we get zero when we project the associated k -current $[F]$ into a subspace P of dimension k or less. For, every k -form ϕ on such a subspace is exact, making the pull-back $\pi^*\phi$ likewise exact. The projected current $\pi_{\#}[F] = [\pi \circ F]$ then vanishes because an exact k -form on a compact k -manifold always integrates to zero.

Examples like this show that the support of a current $[\pi \circ F]$ can be empty, while the image of the mapping $\pi \circ F(M)$ is simultaneously large. In particular, the support of the current induced by a k -chain certainly need not *coincide* with the image of that k -chain. But the two sets *will* coincide when M is compact and $F:M \rightarrow N$ is *injective*. This fact plays a key role in our work, because it connects an analytic property of the current $[F]$ to the topology of F itself. We state and prove it as follows:

Lemma 3.8. *When M is compact and $F:M \rightarrow N$ is an injective lipschitz k -chain, we have $\text{spt}([F]) = F(M)$.*

Proof. When M is compact, $F(M)$ is closed. So any point $p \notin F(M)$ has a neighborhood separating it from $F(M)$, and, clearly, excluding p from $\text{spt}([F])$. This implies half of our lemma: *When M is compact and $F:M \rightarrow N$ is a lipschitz k -chain, we have $\text{spt}([F]) \subset F(M)$.*

Though injectivity was not used above, it is crucial for the reverse inclusion $F(M) \subset \text{spt}([F])$. The latter holds trivially when $k = 0$, so we proceed by induction: Assuming the inclusion in dimension $k - 1$, we argue that it also holds in dimension k .

Suppose, toward a contradiction, that it failed for some k -chain $F : M^k \rightarrow \mathbf{R}^{n+1}$. Then for some $x \in M$, $F(x) \notin \text{spt}([F])$. Since $\text{spt}([F])$ is closed and F is continuous, we then get a non-empty open ball $B \subset M$ with

$$x \in B \quad \text{and} \quad F(\bar{B}) \cap \text{spt}([F]) = \emptyset. \quad (2)$$

On the one hand, this implies that

$$[F|_B] = 0. \quad (3)$$

Indeed, since M is compact, our injectivity assumption makes F an open map. So for some open $U \subset N$, we have

$$F(M) \cap U = F(B).$$

If $[F|_B]$ did not vanish, this would imply that for some k -form ϕ supported in U , we have $[F](\phi) = [F|_B](\phi) \neq 0$, contradicting Eq. (2) above.

On the other hand, once Eq. (3) holds, we can deduce $[F|_{\partial B}] = \partial[F|_B] = 0$, because the boundary operator commutes with proper lipschitz mappings (Eq. (1) above). It then follows that $\text{spt}([F|_{\partial B}]) = \emptyset \not\supset F(\partial B)$. Since ∂B is a $(k - 1)$ -dimensional sphere, and $F|_{\partial B} : \partial B \rightarrow N$ is a lipschitz injection, this contradicts our induction hypothesis. \square

4. Ovaloids

Our main theorem governs lipschitz cycles on ovaloids.

Definition 4.1. A C^2 hypersurface $Q \subset \mathbf{R}^{n+1}$ is an *ovaloid* if it bounds a compact, convex domain. We call an ovaloid *strict* if it is strictly convex, i.e., when its outward unit normal (gauss) mapping

$$v : Q \rightarrow \mathbf{S}^n$$

is a diffeomorphism. Equivalently, Q is strict when it has everywhere positive principal curvatures.

When an ovaloid Q is symmetric with respect to reflection across a hyperplane $P \subset \mathbf{R}^{n+1}$, the symmetry induces a smooth involution $\rho : Q \rightarrow Q$. Choosing a unit vector u normal to P , we can express this symmetry by the formula

$$\rho(x) = x - 2(x \cdot u)u. \quad (4)$$

In this situation, the symmetry hyperplane meets Q along the zero set of $x \cdot u$. The latter, a smooth hypersurface in Q which we call an *equator*, forms the fixed-point set of ρ , and separates Q into two open topological discs that we call *hemispheres*.

Though a general ovaloid $Q \subset \mathbf{R}^{n+1}$ has no such symmetry, *strict* ovaloids enjoy a very similar involution relative to any hyperplane P in \mathbf{R}^{n+1} . Just as in the symmetric case, this involution exchanges two hemispherical discs in Q while fixing the smooth “equator” that separates them. We can define it very conveniently using Steiner symmetrization:

Definition 4.2. Suppose $K \subset \mathbf{R}^{n+1}$ is a compact convex domain bounded by an ovaloid Q . Each line perpendicular to a fixed hyperplane $P \subset \mathbf{R}^{n+1}$ intersects K in a closed segment (possibly one point or empty). Translating every such segment along the line containing it until its midpoint lies on P , we map K to a new set which is clearly symmetric across P . The boundary of this set again forms an ovaloid [3, Theorem 1.2.1], which we label Q_P , and refer to as the *Steiner P -symmetral* of Q . The resulting map

$$\sigma : Q \rightarrow Q_P$$

is clearly continuous and injective, hence (by compactness of Q) a homeomorphism. We call it the *Steiner P -symmetrization* of Q .

We now use σ to construct the promised involution on Q .

Definition 4.3. The *P -involution* on a strict C^2 ovaloid Q is the map

$$\rho := \sigma^{-1} \circ \rho_P \circ \sigma, \quad (5)$$

where ρ_P denotes reflection across P (Eq. (4)). We call the fixed-point set

$$\Gamma := \{x \in Q : \rho(x) = x\}$$

the *P -equator* of Q .

Lemma 4.4. *Given any hyperplane P through the origin in \mathbf{R}^{n+1} , we have*

- (1) *The P -equator Γ on a strict C^2 ovaloid $Q \subset \mathbf{R}^{n+1}$ comprises those points of Q having unit normals in P :*

$$\Gamma = v^{-1}(P \cap \mathbf{S}^n).$$

- (2) *Γ is C^1 diffeomorphic to \mathbf{S}^{n-1} , and*

- (3) *Γ splits its complement in Q into two topological hemispheres, each forming the graph of a C^2 function over a common open set in P .*

Proof. A line perpendicular to P can intersect a strict ovaloid Q in only two ways: Either it (a) grazes Q tangentially, in which case our involution ρ fixes the point of contact, or (b) it pierces Q transversally at exactly two points, and ρ swaps this pair. By definition, the P -equator $\Gamma \subset Q$ comprises the points of case (a). So when $q \in \Gamma$, $T_q Q$ contains a line perpendicular to P , and we have $v(q) \in P$. This proves the lemma's first statement, and since the unit normal map on a strict C^2 ovaloid is a C^1 diffeomorphism, the second statement follows easily.

Moreover, we see that each of the hemispherical regions complementing Γ in Q now comprises points q with $v(q) \notin P$. By the Inverse Function Theorem, this ensures that the orthogonal projection $\pi : \mathbf{R}^{n+1} \rightarrow P$ restricts to a local C^2 diffeomorphism on each hemisphere. Since π also injects on each hemisphere into P , it now follows that both hemispheres are C^2 graphs over the region bounded by $\pi(\Gamma) \subset P$. This verifies conclusion (3). \square

We need one additional characterization of ρ and Γ .

Lemma 4.5. *We can express ρ using a continuous “signed P -height” function $\lambda: Q \rightarrow \mathbf{R}$ via*

$$\rho(x) = x - 2\lambda(x)u. \quad (6)$$

Here u denotes a unit vector normal to P , and we have $\Gamma = \lambda^{-1}(0)$.

Proof. Using σ to pull back Eq. (4) (with ρ replaced by ρ_P there) from Q_P , we get

$$\rho(x) = \sigma^{-1}(\sigma(x) - 2(\sigma(x) \cdot u)u).$$

Since σ acts by simple translation on the u -parallel line through x , this immediately gives (6), with

$$\lambda(x) = \sigma(x) \cdot u.$$

Continuity of σ makes λ continuous too. \square

5. Main results

We now combine our observations about lipschitz chains and ovaloids to produce a basic technical result needed by our main theorem. As above, π denotes orthogonal projection onto a hyperplane $P \subset \mathbf{R}^n$, and ρ is the P -involution of Definition 4.3.

Proposition 5.1. *Suppose we inject a compact, oriented riemannian manifold M^{n-1} into a strict C^2 ovaloid $Q \subset \mathbf{R}^{n+1}$ using a lipschitz map F . If $\pi_{\#}[F] = 0$, then ρ maps $F(M)$ to itself, and reverses orientation; that is, $F^{-1} \circ \rho \circ F$ defines an orientation-reversing homeomorphism of M .*

Proof. We first note that $F(M)$ cannot lie wholly in the P -equator Γ . If it did, the injectivity of π on Γ would make $\pi \circ F$ injective on M , and imply, by Lemma 3.8, that $\text{spt}(\pi_{\#}[F]) = \pi(F(M)) \neq \emptyset$. This violates our assumption that $\pi_{\#}[F] = 0$.

We now argue that ρ leaves $F(M)$ setwise fixed, as the proposition claims. Indeed, suppose not. Then since $F(M)$ does not lie wholly in Γ , we must have some point q^+ in one of the hemispheres Ω^{\pm} complementing Γ in Q for which

$$q^+ \in F(M), \quad \text{but} \quad q^- := \rho(q^+) \notin F(M). \quad (7)$$

Without loss of generality, assume $q^+ \in \Omega^+$. Then q^- lies in Ω^- , but, missing the compact set $F(M)$, it must lie in an open set $U^- \subset \Omega^-$ likewise disjoint from $F(M)$.

Define $U^+ := \rho(U^-) \subset \Omega^+$. Between the third conclusion of Lemma 4.4 and our definition of ρ , we then see that both U^+ and U^- are graphs of C^2 functions over a common open subset of $U \subset P$. In other words,

$$\pi(U^+) = \pi(U^-) =: U \subset P.$$

Now observe that any 1-form ϕ supported in U pulls back to Q , via π , as the sum of 1-forms ϕ^+ and ϕ^- supported in U^+ and U^- respectively:

$$\pi^*\phi = \phi^+ + \phi^-, \quad \text{with} \quad \phi^+ = (\pi|_{U^+})^*\phi \quad \text{and} \quad \phi^- = (\pi|_{U^-})^*\phi.$$

Since $\pi|_{U^+}$ is a diffeomorphism, the relation $\phi \leftrightarrow \phi^+$ induces an isomorphism $\mathcal{D}^1(U) \approx \mathcal{D}^1(U^+)$. In view of Eq. (7) above, Lemma 3.8 shows that the support of $[F]$ contains $q^+ \in U^+$. So there

exists a 1-form ϕ supported in U such that $[F](\phi^+) \neq 0$. On the other hand, $[F](\phi^-)$ does vanish, because the support of ϕ^- lies in U^- , which, by construction, entirely misses $\text{spt}([F])$. But then

$$(\pi_{\#}[F])(\phi) = [F](\pi^*\phi) = [F](\phi^+ + \phi^-) = [F](\phi^+) \neq 0.$$

This contradicts the vanishing of $\pi_{\#}[F]$, and thereby confirms our lemma's first conclusion: ρ preserves $F(M)$.

It also shows that the formula

$$\Phi := F^{-1} \circ \rho \circ F$$

constructs a well-defined mapping of M . Since both F and ρ are injective, and M is compact, Φ is bicontinuous. It remains to show that it reverses orientation.

We started by ruling out the inclusion $F(M) \subset \Gamma$. But $F(M)$ cannot avoid Γ entirely, as this would place it completely inside either Ω^+ , or Ω^- , where it could not be preserved by ρ . Hence $F^{-1}(\Gamma)$, which comprises the fixed-point set of Φ , is not empty. On the other hand ρ swaps Ω^+ and Ω^- , so Φ moves every component of $M \setminus F^{-1}(\Gamma)$. The brief note by Brown and Kister [1] now shows that there are just two such components, which Φ must swap while reversing orientation. \square

The main result discussed in our introduction, Theorem 5.3, actually follows as a corollary of the more complex statement below:

Theorem 5.2. *Consider a compact embedded lipschitz $(n-1)$ -cycle $F:M \rightarrow Q$ on a strict C^2 ovaloid $Q \subset \mathbf{R}^{n+1}$. If $[F]$ projects to zero on $n+1$ independent hyperplanes, then M admits a fixed-point-free homeomorphism of degree $(-1)^{n-1}$.*

Proof. We argue by contradiction: Suppose that for some compact, embedded lipschitz $(n-1)$ -cycle $F:M \rightarrow Q$, we had $n+1$ independent hyperplanes P_i whose corresponding orthogonal projections $\pi_i:\mathbf{R}^{n+1} \rightarrow P_i$ all annihilated $[F]$, i.e.,

$$\pi_{i\#}[F] = 0, \quad i = 1, 2, \dots, n+1.$$

Choose a unit normal vector u_i for each P_i . Since π_i commutes with translation along u_i , we may assume each hyperplane P_i passes through the origin. Lemma 4.5 now assigns a P_i -involution ρ_i , along with a signed P_i -height function λ_i , to Q for each of these hyperplanes, so that for each $i = 1, \dots, n+1$, we have

$$\rho_i(x) = x - 2\lambda_i(x)u_i \quad \text{for all } x \in Q. \quad (8)$$

Since $\pi_{i\#}[F] = 0$ for each i , we also know, by Proposition 5.1, that each of the homeomorphisms

$$\Phi_i := F^{-1} \circ \rho_i \circ F$$

reverses orientation on M . The composition

$$\begin{aligned} \Psi &:= \Phi_{n+1} \circ \Phi_n \circ \dots \circ \Phi_2 \circ \Phi_1 \\ &= F^{-1} \circ \rho_{n+1} \circ \rho_n \circ \dots \circ \rho_2 \circ \rho_1 \circ F, \end{aligned}$$

therefore preserves or reverses the orientation of M depending, respectively, on whether n is odd or even. In other words, $\Psi:M \rightarrow M$ has degree $(-1)^{n+1} = (-1)^{n-1}$, and we can complete the proof by showing that Ψ fixes no point in M .

Indeed, if we *had* a fixed point $x \in M$, then $y = F(x) \in Q$, would satisfy

$$\rho_{n+1} \circ \rho_n \circ \cdots \circ \rho_2 \circ \rho_1(y) = y, \quad \text{with} \quad y = F(x).$$

Expand this out using Eq. (8), cancel the lone y on each side, and divide by -2 , to get

$$\lambda_1(y)u_1 + \lambda_2(\rho_1(y))u_2 + \lambda_3(\rho_2(\rho_1(y))) + \cdots + \lambda_{n+1}(\rho_n(\rho_{n-1}(\cdots(\rho_1(y))\cdots)))u_{n+1} = 0.$$

Since the u_i 's are linearly independent, their coefficients above must all vanish. Working recursively from left to right using the last conclusion of Lemma 4.5 and the fact that ρ_i fixes Γ_i , we now reason as follows:

$$\lambda_1(y) = 0 \Rightarrow y \in \Gamma_1 \quad \text{and} \quad \rho_1(y) = y.$$

Knowing now that $\rho_1(y) = y$, we subsequently get

$$\lambda_2(y) = \lambda_2(\rho_1(y)) = 0 \Rightarrow y \in \Gamma_2 \quad \text{and} \quad \rho_2(y) = y.$$

Continuing in this way, we find that for *all* $i = 1, 2, \dots, n+1$, we have $\lambda_i(y) = 0$ and $y \in \Gamma_i$. It then follows from the first conclusion of Lemma 4.4 consequently that

$$y \in \Gamma_1 \cap \Gamma_2 \cap \cdots \cap \Gamma_{n+1} = v^{-1}(S^{n+1} \cap P_1 \cap P_2 \cap \cdots \cap P_{n+1}).$$

But the intersection on the right is empty, because $n+1$ independent hyperplanes in \mathbf{R}^{n+1} intersect only at the origin. We have thus contradicted the existence of a fixed point for Ψ , and proven the theorem. \square

Our main result now follows easily:

Theorem 5.3. *A lipschitz embedding of a homology $(n-1)$ -sphere on a strict C^2 ovaloid in \mathbf{R}^{n+1} cannot project to zero on $n+1$ independent hyperplanes.*

Proof. By Theorem 5.2, a violation of this theorem would produce a fixed-point-free homeomorphism of an $(n-1)$ -dimensional homology sphere with degree $(-1)^{n-1}$. By a well-known consequence of the Lefschetz fixed-point theorem [5, Corollary 6.21], no such map exists. \square

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